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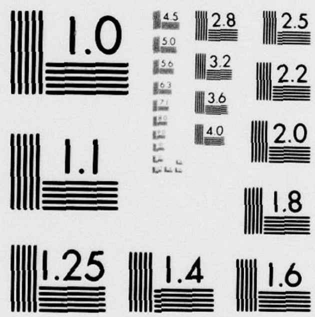

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(6) ON TESTING EQUALITY OF TWO EXPONENTIAL DISTRIBUTIONS UNDER COMBINED TYPE II CENSORING.

by

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ON TESTING EQUALITY OF TWO EXPONENTIAL DISTRIBUTIONS UNDER COMBINED TYPE II CENSORING

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ON TESTING EQUALITY OF TWO EXPONENTIAL DISTRIBUTIONS
UNDER COMBINED TYPE II CENSORING

G. K. Bhattacharyya and Kishan G. Mehrotra

ABSTRACT

The problem of testing equality of the scale parameters of two exponential distributions is considered under a combined sample type II censoring scheme that has been extensively treated in non-parametric inference. Sufficiency and invariance considerations lead to a pair of statistics, namely, the proportions of failure count and the total time on test of the first sample to those of the combined sample. Exact distribution and moment properties are discussed, and asymptotic joint normality under local alternatives is established in a general framework. The invariant test that maximizes the Pitman efficacy is seen to be based on the difference between these two proportions, and is asymptotically equivalent to the likelihood ratio test.

KEY WORDS: Exponential distribution; Censored data; Invariant tests; Asymptotic relative efficiency.

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1. INTRODUCTION

The exponential distribution as a model for failure times, and censored sampling as a basis for statistical inference both play major roles in life testing and reliability studies. In a comparative study of two types of components, suppose the life distributions are assumed to be exponential with unknown scale parameters θ_1 and θ_2, and one wishes to test the null hypothesis $H_0 : \theta_1 = \theta_2$ from the failure time data of a number of components of each type. If the samples are fully observed, the standard theory of exponential families provides the uniformly most powerful (UMP) unbiased test which is also UMP invariant. Only minor modifications in the derivation are needed to handle the situation where the samples are independently right censored at specified order statistics (separate type II censoring). In either case, the optimal test is based on the ratio of the total times on test of the two groups of components.

Another important design of censored life tests consists in simultaneously observing two groups of components and recording the successive failure times as they occur until a prespecified order statistic of the combined sample is observed. For such jointly type II censored data, extensive literature exists in regard to nonparametric tests, for instance, Rao, Savage and Sobel (1960), Johnson and Mehrotra (1972), Johnson (1974). A nonparametric test is also proposed by Halperin (1960) for jointly

type I censored data and later extended by Gehan (1965) to more general types of time censoring.

Curiously, the development of two-sample tests under joint type II censoring has been confined to the area of nonparametric analysis whereas the exponential model is extensively treated under separate type II censoring; see for example, Epstein and Sobel (1953), Epstein and Tsao (1953), Perng (1977). The object of the present article is to study the distributions and asymptotic power properties of some invariant tests for testing equality of two exponential scale parameters under a jointly type II censoring scheme.

A reduction of the testing problem by sufficiency and invariance results in a pair of statistics, namely, the proportions of failure count and the total time on test of the first sample to those of the combined sample. The null distribution and moments of these statistics are derived in Section 3. The test based on failure count provides a simple test whose power is also easy to calculate and its usefulness extends beyond the exponential model. On the other hand, because the total time on test statistic provides the UMP unbiased tests in complete samples as well as separate type II censored samples from exponential populations, its performance in the case of joint type II censoring is of natural interest. Interestingly, the joint censoring scheme gives rise to considerable complexity in its distribution theory even under the null hypothesis.

Some general results on the asymptotic distributions of these statistics under the null hypothesis as well as local alternatives are outlined in Section 4 and then specialized to the exponential model. The technical details are given in the Appendix. Section 5 contains comparisons of these tests with the locally most powerful rank test in light of Pitman asymptotic relative efficiency (ARE). An invariant test that maximizes the ARE is found to involve the difference between the two aforementioned test statistics and is asymptotically equivalent to the likelihood ratio test.

2. LIKELIHOOD FUNCTION AND MAXIMAL INVARIANTS

Let $X_1,\dots,X_m$ and $Y_1,\dots,Y_n$ be independent random samples from absolutely continuous distribution functions F and G respectively, and let the combined sample order statistics be denoted by $W_1 \le \dots \le W_N$ where $N = n+m$. Under the joint type II censoring scheme, only a specified number $r\ (<N)$ of the combined sample order statistics are observed and the source of each is also noted. Thus, the observable data consist of $(\underset{\sim}{Z},\underset{\sim}{W})$ where $\underset{\sim}{Z}' = (Z_1,\dots,Z_r)$, $\underset{\sim}{W}' = (W_1,\dots,W_r)$ and $Z_i = 1(0)$ if W_i is an $X(Y)$. Denoting $\sum_{i=1}^{r} Z_i = M_r$, $\sum_{i=1}^{r} (1-Z_i) = N_r$ where $M_r + N_r = r$, the joint density of $(\underset{\sim}{Z},\underset{\sim}{W})$ is given by

$$P_{F,G}(\underset{\sim}{z},\underset{\sim}{w}) = \frac{m!\,n!}{(m-m_r)!\,(n-n_r)!}\prod_{i=1}^{r}\left\{f^{z_i}(w_i)\,g^{1-z_i}(w_i)\right\}\bar{F}(w_r)^{m-m_r}\,\bar{G}(w_r)^{n-n_r} \qquad (2.1)$$

where f and g are respectively the pdf's of F and G, $\bar{F} = 1-F$, and $\bar{G} = 1-G$. In this article we focus attention on the parametric family of exponential distributions $\bar{F}(x) = \exp(-\theta_1 x)$, $\bar{G}(x) = \exp(-\theta_2 x)$, $x > 0$, where θ_1 and θ_2 are unknown, and consider the problem of testing $H_0: \theta_1 = \theta_2$ vs $H_1: \theta_2 > \theta_1$. The density (2.1) then reduces to

$$P_{\underset{\sim}{\theta}}(\underset{\sim}{z},\underset{\sim}{w}) = \frac{m!\,n!}{(m-m_r)!\,(n-n_r)!}\,\theta_1^{m_r}\,\theta_2^{n_r}\exp(-u_1\theta_1 - u_2\theta_2) \qquad (2.2)$$

where

$$U_1 = \sum_{i=1}^{r} Z_i W_i + (m-M_r)W_r = \sum_{i=1}^{M_r} X_{(i)} + (m-M_r)W_r\,,$$

$$U_2 = \sum_{i=1}^{r} (1-Z_i)W_i + (n-N_r)W_r = \sum_{j=1}^{N_r} Y_{(j)} + (n-N_r)W_r\,, \qquad (2.3)$$

and $X_{(\cdot)}$, $Y_{(\cdot)}$ denote the order statistics of the individual samples. The statistics U_1 and U_2 represent the total times on test of the first and the second samples, respectively, when observations are terminated at W_r. The family of densities (2.2) constitutes an exponential family and (M_r, U_1, U_2) are sufficient statistics. Since three components are present in the minimal sufficient statistics while the parameter space is two dimensional, the standard theory of optimal tests for exponential families does not apply. This is analogous to the case of type I censored sample from a single exponential population for which certain reasonable test procedures are discussed in Bain (1978).

Turning to invariance, we note that the testing problem for the exponential model (2.2) remains invariant under the group of common scale transformations $X_i' = cX_i$, $Y_j' = cY_j$, $c > 0$. The induced transformation on the space of sufficient statistics is $M_r' = M_r$, $U_1' = cU_1$ and $U_2' = cU_2$ and, consequently, any invariant test must be a function of the maximal invariants $(M_r, U_1U_2^{-1})$ or equivalently

$$T_1 = \frac{M_r}{r}, \quad T_2 = \frac{U_1}{U_1 + U_2}. \tag{2.4}$$

The statistic T_1 is the proportion of failures in the first sample to the total of r failures while T_2 represents the ratio of the total time on test of the first sample to that of the combined sample. Since under H_0, the distribution of the maximal invariant is free of the nuisance parameter, either of them can be employed to construct a test. The performance of T_2 is of special interest in so far as it is known to provide the UMP unbiased and invariant test in the case of no censoring $(r = N)$. Although T_1 arises as a component of the maximal invariants in the exponential model, it provides a simple distribution-free test known as a generalized quantile test, c.f. Hájek and Šidak (1967), p. 90. With time censored data from a single exponential population, the test based on failure count is noted for its simplicity and satisfactory power properties, and this also points to the need of studying the performance of T_1. The properties of these statistics are investigated in the following sections.

3. DISTRIBUTIONS AND MOMENTS OF T_1 AND T_2

We first consider the distribution of $M_r = rT_1$ in the null case $F = G$. Since M_r is the number of X's among the first r order statistics of the combined sample, and $X_1, \ldots, X_m, Y_1, \ldots, Y_n$ are independent and identically distributed, by a direct combinatorial argument it follows that M_r has the hypergeometric distribution

$$P_{F,F}[M_r = m_r] = \binom{m}{m_r}\binom{n}{n_r}\Big/\binom{N}{r}, \qquad \max(r-n, 0) \le m_r \le \min(m, r). \tag{3.1}$$

Hence the null mean and variance of T_1 are given by

$$E(T_1) = \frac{m}{N}, \quad \operatorname{Var}(T_1) = \frac{mn(N-r)}{rN(N-1)}. \tag{3.2}$$

For the distribution of M_r in non-null case $F \neq G$, one can refer to (2.1) and integrate the rhs over $\underset{\sim}{w}$. An easier approach is to use the event relation

$$[M_r = m_r] = \left[Y_{(n_r)} < X_{(m_r)} < Y_{(n_r+1)}\right] \cup \left[X_{(m_r)} < Y_{(n_r)} < X_{(m_r+1)}\right]. \tag{3.3}$$

Since $P\left[Y_{(n_r)} < x < Y_{(n_r+1)}\right] = \binom{n}{n_r} G^{n_r}(x)\bar{G}^{n-n_r}(x)$, by virtue of independence between $X_{(\cdot)}$ and $Y_{(\cdot)}$ it follows that

$$P\left[Y_{(n_r)} < X_{(m_r)} < Y_{(n_r+1)}\right] = \binom{m}{m_r}\binom{n}{n_r} m_r \int G^{n_r}(x)\bar{G}^{n-n_r}(x) F^{m_r-1}(x)\bar{F}^{m-m_r}(x) f(x)\,dx,$$

and similarly the probability of the second event on the rhs of (3.3) can be obtained. These lead to

$$P_{F,G}[M_r=m_r] = \binom{m}{m_r}\binom{n}{n_r}\int G^{n_r-1}(x)\bar{G}^{n-n_r}(x)F^{m_r-1}(x)\bar{F}^{m-m_r}(x)$$

$$[m_r G(x)f(x) + n_r F(x)g(x)]dx .$$

Considering the Lehmann alternatives where F and G are related by $\bar{G}(x) = \bar{F}^{\Delta}(x)$, the above expression reduces to

$$P_{\Delta}[M_r=m_r] = \binom{m}{m_r}\binom{n}{n_r}\int_0^1\left\{u^{m-m_r+\Delta(n-n_r)}(1-u)^{m_r-1}(1-u^{\Delta})^{n_r-1}\right\}$$

$$\left\{m_r(1-u^{\Delta}) + n_r\Delta u^{\Delta}\right\}du .$$

After straightforward but lengthy calculations, it further simplifies to the following form

$$P_{\Delta}[M_r=m_r] = \binom{m}{m_r}\binom{n}{n_r}[m-m_r+\Delta(n-n_r)]\int_0^1 u^{m-m_r+\Delta(n-n_r)-1}(1-u)^{m_r}(1-u^{\Delta})^{n_r}du . \tag{3.4}$$

Clearly, the distribution (3.4) also holds for the exponential alternatives if we identify $\Delta = \theta_2/\theta_1$. With the aid of expression (3.4), the exact power of the test based on T_1 can be computed.

In contrast with the simple results for T_1, the distribution of T_2 turns out to be considerably involved even in the exponential null case. In order to characterize this distribution we first observe from (2.1) that under $F = G$, the random rectors $\underset{\sim}{W}$ and $\underset{\sim}{Z}$ are independent, and consequently $\underset{\sim}{W}$ is independent of M_r. Given $M_r = m_r$, the vector $\underset{\sim}{Z}$ can have $\binom{r}{m_r}$ realizations which are equally likely. Let $\mathcal{C}$ denote the set of all ordered m_r-tuples

of the integers $\{1,2,\ldots,r\}$. Since $T_2 = \left(\sum_{i=1}^{r} Z_i W_i + (m-M_r)W_r\right)/U$, where $U = \sum_{i=1}^{r} W_i + (N-r)W_r$ is free of $\underset{\sim}{Z}$, the conditional distribution of T_2 given $M_r = m_r$ is the same as that of

$$S_{\underset{\sim}{\delta}(m_r)} = [W_{\delta_1} + \ldots + W_{\delta_{m_r}} + (m-m_r)W_r]/U \tag{3.5}$$

where $\underset{\sim}{\delta}(m_r) = (\delta_1,\ldots,\delta_{m_r})$ is a random vector independent of $\underset{\sim}{W}$ and taking values in the set $\mathcal{C}$, with equal probabilities. Thus, the pdf of T_2 given $M_r = m_r$ can be expressed as

$$f_{T_2|m_r}(t) = \binom{r}{m_r}^{-1} \sum_{\underset{\sim}{c}(m_r)\in\mathcal{C}} f_{\underset{\sim}{c}(m_r)}(t) \tag{3.6}$$

where $f_{\underset{\sim}{c}(m_r)}(t)$ is the pdf of $S_{\underset{\sim}{c}(m_r)}$ and $\underset{\sim}{c}(m_r) = (c_1,\ldots,c_{m_r})$ stands for a typical realization of $\underset{\sim}{\delta}(m_r)$.

Further assuming that $\bar{F}(x) = \bar{G}(x) = \exp(-\theta x)$, the normalized spacings $D_j \equiv (N-j+1)(W_j - W_{j-1})$, $j = 1,2,\ldots,r$, are iid exponential and we have the relations

$$W_i = \sum_{j=1}^{i}(N-j+1)^{-1}D_j , \quad U = \sum_{j=1}^{r} D_j . \tag{3.7}$$

Letting $D_j^* = U^{-1}D_j$ the vector $(D_1^*,\ldots,D_{r-1}^*)$ has a Dirichlet distribution with all parameters equal to 1, that is, with uniform mass on the region $\left(0 < \sum_{j=1}^{r-1} D_j^* < 1,\ D_j^* > 0,\ j = 1,\ldots,r-1\right)$. Consequently, the distribution of $S_{\underset{\sim}{c}(m_r)}$ is given by the structure

$$S_{\underset{\sim}{c}(m_r)} = (m-m_r)\sum_{j=1}^{r}(N-j+1)^{-1}D_j^* + \sum_{k=1}^{m_r}\sum_{j=1}^{c_k}(N-j+1)^{-1}D_j^*$$

$$= \sum_{j=1}^{r} d_j(\underset{\sim}{c}(m_r))D_j^*$$

where $d_j(\underset{\sim}{c}(m_r)) = (N-j+1)^{-1}(m-k+1)$ for $c_{k-1} < j \le c_k$, $k = 1,\dots,m_r$, $c_0 = 0$. Thus, $f_{\underset{\sim}{c}(m_r)}$ is the pdf of a linear function of Dirichlet random variables. This representation shows that in the scheme of jointly censored data, the null distribution of the total time on test statistic T_2 has a rather complex structure unlike that in the scheme of separate censoring.

Even though the above characterization of the distribution of T_2 does not yield a simple expression for its pdf, it does facilitate the calculations of moments. Let

$$\alpha_j = (N-j+1)^{-1},\quad \beta_j = \sum_{i=1}^{j}\alpha_i,\quad \tau_j = \sum_{i=1}^{j}\alpha_i^2,\quad j = 1,2,\dots,N. \tag{3.8}$$

In Lemma 1 below we establish some identities relating α_j's, β_j's and τ_j's which will be used in the calculations of moments of T_2.

<u>Lemma 1</u>: For $r \le N$, the following identities hold.

$$\sum_{i=1}^{r}\tau_i + (N-r)\tau_r = \beta_r \tag{3.9}$$

$$\sum_{i=1}^{r}\beta_i^2 + (N-r)\beta_r^2 = 2r - (2N-2r+1)\beta_r. \tag{3.10}$$

<u>Proof</u>: By interchanging the order of summation in $\sum_{i=1}^{r}\sum_{j=1}^{i}\alpha_j^2$ and using the relation $(r-j+1)\alpha_j = 1-(N-r)\alpha_j$, we have

$$\sum_{i=1}^{r}\tau_i = \sum_{j=1}^{r}(r-j+1)\alpha_j^2$$

$$= \sum_{j=1}^{r}\alpha_j - (N-r)\sum_{j=1}^{r}\alpha_j^2 = \beta_r - (N-r)\tau_r,$$

which establishes the first identity. In a similar manner, letting $\Sigma_{jj',i}$ denote the operation of summing over $1 \le j < j' \le i$, for a fixed i, we get

$$\sum_{i=1}^{r}\sum\nolimits_{jj',i}\alpha_j\alpha_{j'} = \sum\nolimits_{jj',r}(r-j'+1)\alpha_j\alpha_{j'}$$

$$= \sum_{j=1}^{r}(r-j)\alpha_j - (N-r)\sum\nolimits_{jj',r}\alpha_j\alpha_{j'}. \tag{3.11}$$

Expanding $\left(\sum_{j=1}^{i}\alpha_j\right)^2$ involved in $\sum_{i=1}^{r}\beta_i^2$ and then using (3.9) and (3.11) we obtain

$$\sum_{i=1}^{r}\beta_i^2 = \sum_{i=1}^{r}\tau_i + 2\sum_{i=1}^{r}\sum\nolimits_{jj',i}\alpha_j\alpha_{j'}$$

$$= \beta_r + \sum_{j=1}^{r}(r-j)\alpha_j - (N-r)\left\{\tau_r + 2\sum\nolimits_{jj',r}\alpha_j\alpha_{j'}\right\}.$$

The result (3.10) follows because $\tau_r + 2\Sigma_{jj',r}\alpha_j\alpha_{j'} = \beta_r^2$. ▮

<u>Lemma 2</u>: If $\bar{F}(x) = \bar{G}(x) = \exp(-\theta x)$, $x > 0$, then

$$E(T_2) = \frac{m}{N}, \quad Var(T_2) = \frac{mn}{N(N-1)(r+1)}\left[\frac{2N-r-1}{N} - \frac{2(N-r)}{r}\beta_r\right]. \tag{3.12}$$

<u>Proof</u>: Since $\underset{\sim}{W}$ and $\underset{\sim}{Z}$ are independent and $E(Z_i) = N^{-1}m$, the conditional expectation of U_1 given $\underset{\sim}{W}$ is $N^{-1}mU$ and hence $E(T_2|\underset{\sim}{W}) = N^{-1}m$. This establishes the first result. To calculate $E(T_2^2)$, it is again convenient to start with its conditional expectation given $\underset{\sim}{W}$. From the distribution of $\underset{\sim}{Z}$ it immediately follows that $E(Z_i^2) = N^{-1}m$ and $E(Z_iZ_j) = [N(N-1)]^{-1}m(m-1)$ for $i \neq j$. Now letting $Q_i = (W_i - W_r)U^{-1}$, we have $T_2 = \sum_{i=1}^{r} Z_iQ_i + mU^{-1}W_r$ where the second term is free of $\underset{\sim}{Z}$. Using the above moments of Z_i's, an expression of $E(T_2^2|\underset{\sim}{W})$ is obtained in terms of W_r and Q_i's which, along with the relations

$$\sum_{i=1}^{r} Q_i = (1 - U^{-1}W_rN),$$

$$\sum_{i=1}^{r} Q_i^2 = U^{-2}\left[\sum_{i=1}^{r-1} W_i^2 - 2W_r\sum_{1}^{r-1} W_i + (r-1)W_r^2\right]$$

simplifies to

$$E(T_2^2|\underset{\sim}{W}) = \frac{m(m-1)}{N(N-1)} + \frac{mn}{N(N-1)}\left\{\sum_{i=1}^{r-1}\left(\frac{W_i}{U}\right)^2 + (N-r+1)\left(\frac{W_r}{U}\right)^2\right\}. \tag{3.13}$$

Now referring to (3.7) note that $U^{-1}W_i = \sum_{j=1}^{i} \alpha_j D_j^*$ where $(D_j^*, j = 1,\ldots,r-1)$ have a joint Dirichlet distribution. Since each D_j^* is distributed as a beta $(1,r-1)$, we have $E(D_j^*) = r^{-1}$,

$E(D_j^{*2}) = 2[r(r+1)]^{-1}$. Also, since the distribution of $(D_j^* + D_{j'}^*)$ is beta $(2,r-2)$, $E(D_j^* + D_{j'}^*)^2 = 6[r(r+1)]^{-1}$ and we obtain $E(D_j^*D_{j'}^*) = [r(r+1)]^{-1}$, for $j \neq j'$. Consequently $E(W_i/U) = r^{-1}\beta_i$ and $E(W_i^2/U^2) = [r(r+1)]^{-1}(\tau_i + \beta_i^2)$ where τ_i and β_i are given in (3.8). From (3.13) we then obtain

$$E(T_2^2) = m(m-1)[N(N-1)]^{-1}\Big[1 + [r(r+1)]^{-1}\Big\{\sum_{i=1}^{r-1}(\tau_i + \beta_i^2) + (N-r)(\tau_r + \beta_r^2)\Big\}\Big],$$

and finally, an application of Lemma 1 leads to the second expression in (3.12). ▌

Note that T_1 and T_2 have the same null expectations. Following the approach of Lemma 2 we also obtain

$$Cov(T_1, T_2) = \frac{mn(N-r)}{N^2(N-1)}\left(1 - \frac{N}{r}\beta_r\right)$$

which is clearly a negative quantity. The exact null distribution of the total time on test statistic T_2 is complicated because of the combined sample joint censoring and the consequent dependence of U_1 and U_2 on the common W_r. If the censoring points were determined by specified order statistics of the individual samples, as in the case of separate type II censoring, T_2 would have an exact beta distribution. This suggests that a beta distribution can serve as a reasonable approximation in the present case. The moments obtained in Lemma 2 can be used to

determine the parameters of the approximating distribution. A beta approximation is further reinforced by a moment property (3.14) established below.

Some moment relations among M_r, U_1 and U_2 can be obtained by exploiting the likelihood function (2.2) and taking expectation of the first derivatives of its logarithm with respect to θ_1 and θ_2. In particular, for $\theta_1 = \theta_2 = \theta$, we obtain

$$E(M_r) = \theta E(U_1), \quad E(r - M_r) = \theta E U_2 .$$

Since in the null case $\theta E(U_1 + U_2) = r$, an interesting consequence of (3.12) and the above equalities is

$$E\left(\frac{U_1}{U_1 + U_2}\right) = \frac{E(U_1)}{E(U_1 + U_2)} . \tag{3.14}$$

This relation between the expectation of a ratio and ratio of the expectations is well known for independent gamma random variables in which case the ratio has a beta distribution. This property is enjoyed by T_2 even though U_1 and U_2 are not independent. In fact, they are not even asymptotically independent as can be seen in course of the proof of Theorem 1.

4. ASYMPTOTIC THEORY

Our object here is to obtain the asymptotic joint distribution of the invariant statistics T_1 and T_2 under a sequence of local alternatives in the exponential model. Unlike separate censoring, the combined type II censoring scheme complicates the asymptotic results even for the exponential case. It turns out that our derivations can be geared for a general class of life distributions without any added complexity. The general results are stated in Theorem 1 and then specialized to the exponential case in Corollary 1.

Let $\{F_\Delta, \Delta \in \Theta \subseteq R\}$ be a real parameter family of absolutely continuous cdf's of non-negative random variables, Δ_0 a value specified by the null hypothesis, and $\Delta_N = \Delta_0 + N^{1/2} h_N$ a sequence of local alternatives such that as $N \to \infty$, $h_N \to h$, finite. Consider independent random samples $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ from the cdf's F_{Δ_0} and F_{Δ_N}, respectively, and denote by W_r the combined sample r-th order statistic where $r = [Np]$, the integer part of Np, $0 < p < 1$, $N = m+n$. Denoting the empirical cdf's of $(X_1, \ldots, X_m)$ and $(Y_1, \ldots, Y_n)$ by F_{1m} and F_{2n}, respectively, and letting $\lambda_N = N^{-1} m$, $H_N = \lambda_N F_{1m} + (1-\lambda_N) F_{2N}$, the ramdom variables M_r, U_1 and U_2 have the representations

$$\begin{aligned} m^{-1} M_r &= F_{1m}(W_r) \\ m^{-1} U_1 &= \int_0^{W_r} x\,dF_{1m} + W_r \bar{F}_{1m}(W_r) \\ n^{-1} U_2 &= \int_0^{W_r} x\,dF_{2n} + W_r \bar{F}_{2n}(W_r) \end{aligned} \tag{4.1}$$

where $\int_a^b$ is to be interpreted as integral over the interval $(a,b]$. We denote by ζ the p-quantile of F_{Δ_0} and let f_Δ be the pdf of F_Δ. For notational simplicity, the symbols F and f without subscripts will be used in place of F_{Δ_0} and f_{Δ_0}, respectively.

Assumptions (A).

(A1) $F_\Delta(x)$ viewed as a function of (Δ,x) has continuous first partial derivatives in an open neighborhood $\mathcal{C}_0$ of (Δ_0,ζ).

(A2) $f'_\Delta(x) = (\partial/\partial x) f_\Delta(x)$ is bounded in $\mathcal{C}_0$.

(A3) At Δ_0, $\int_0^\zeta F_\Delta(x)dx$ can be differentiated with respect to Δ under integration.

(A4) As $N \to \infty$, $(\lambda_N - \lambda) = o(N^{-1/2})$, $0 < \lambda < 1$.

While discussing asymptotic distributions we will write T_{1N} and T_{2N} in place of T_1 and T_2 and denote $\underset{\sim}{T}'_N = (T_{1N}, T_{2N})$. Let

$$\begin{aligned}
&\overset{\circ}{F}(x) = \frac{\partial}{\partial \Delta} F_\Delta(x)\Big|_{\Delta_0}, \quad \mu_\zeta = \int_0^\zeta x dF(x), \quad q = (1-p) = \bar{F}(\zeta) \\
&\sigma_{11} = pq, \quad \sigma_{12} = q\mu_\zeta, \quad \sigma_{22} = \int_0^\zeta x^2 dF(x) - \mu_\zeta^2 \\
&\underset{\sim}{\Sigma} = (\sigma_{ij}), \quad \nu = \mu_\zeta + \zeta q \\
&\mu_1 = p^{-1}\overset{\circ}{F}(\zeta), \quad \mu_2 = -\nu^{-1}\int_0^\zeta \overset{\circ}{F}(x)dx, \quad \underset{\sim}{\mu}' = (\mu_1, \mu_2).
\end{aligned} \tag{4.2}$$

The limiting distribution of $\underset{\sim}{T}_N$ is stated in Theorem 1 and the proof is outlined in the Appendix.

Theorem 1: Under Assumptions (A) and the aforementioned notations, the limiting distribution of $N^{1/2}(T_{1N}-\lambda, T_{2N}-\lambda)$ under local alternatives Δ_N is bivariate normal $\eta_2(-\lambda(1-\lambda)h\underset{\sim}{\mu}, \lambda(1-\lambda)\underset{\sim}{\tau})$ where

$$\underset{\sim}{\tau} = \begin{pmatrix} p^{-2}\sigma_{11} & (p\nu)^{-1}(\sigma_{12}-\zeta\sigma_{11}) \\ (p\nu)^{-1}(\sigma_{12}-\zeta\sigma_{11}) & \nu^{-2}(\sigma_{22}-2\zeta\sigma_{12}+\zeta^2\sigma_{11}) \end{pmatrix}. \tag{4.3}$$

We now turn to the parametric model involving a pair of exponential distributions with scale parameters θ_1 and θ_2. Considering the reparameterization $(\theta_1,\theta_2) \leftrightarrow (\theta_1, \theta_1\Delta)$ where $\Delta = \theta_2\theta_1^{-1}$, the null hypothesis is $\Delta = 1$. Since by scale invariance, the distribution of (T_{1N}, T_{2N}) is free of θ_1, we can take $\theta_1 = 1$ without loss of generality. Therefore, the formulation of this section is applicable to the special case $\bar{F}_\Delta(x) = \exp(-\Delta x)$, $x > 0$, with $\Delta_0 = 1$. Assumptions (A) are satisfied and after some simplifications the quantities involved in (4,2) reduce to

$$\begin{aligned}
&\zeta = -\ln q, \quad \nu = p, \quad \underset{\sim}{\mu}' = p^{-1}(\zeta q, \zeta q - p) \\
&\underset{\sim}{\tau} = p^{-2}\begin{pmatrix} pq & q(p-\zeta) \\ q(p-\zeta) & p(2-p) - 2\zeta q \end{pmatrix}
\end{aligned} \tag{4.4}$$

Corollary 1: Given that X_i's are iid exponential with pdf $\exp(-x)$ and Y_j's are iid exponential with pdf $\Delta_N^{-1}\exp(-\Delta_N x)$ where $\Delta_N = 1 + h_N N^{-1/2}$ and $h_N \to h$ as $N \to \infty$, the limiting distribution of $N^{1/2}[\underset{\sim}{T}_N - (\lambda,\lambda)']$ is $\eta_2(-\lambda(1-\lambda)h\underset{\sim}{\mu}, \lambda(1-\lambda)\underset{\sim}{\tau})$ where $\underset{\sim}{\mu}$ and $\underset{\sim}{\tau}$ are given in (4.4).

5. CONSIDERATIONS OF PITMAN ARE

In Section 2 it was seen that in the exponential testing problem, T_1 and T_2 together form the maximal invariants. A test based on T_1 alone is of particular interest because its null distribution is extremely simple, a manageable expression could be obtained for its exact non-null distribution and more importantly, the test is distribution-free. On the other hand a test based on the total time on test statistic T_2 is of interest because it provides the UMP unbiased as well as UMP invariant test in the uncensored and independently censored situations. In this section we first compare T_1 and T_2 in terms of their Pitman asymptotic relative efficiency (ARE).

The asymptotic normality of T_{1N} and T_{2N}, under a sequence of local alternatives follows from Corollary 1, and from the expressions of their asymptotic means and variances, the efficacies of T_1 and T_2 are obtained as

$$\begin{aligned} e(T_1) &= \lambda(1-\lambda)h^2\zeta^2 qp^{-1} \\ e(T_2) &= \lambda(1-\lambda)h^2(\zeta q-p)^2[p(2-p)-2\zeta q]^{-1} . \end{aligned} \tag{5.1}$$

The ARE of T_1 vs T_2, given by the ratio $e(T_1)/e(T_2) = \zeta^2 q \times [p(2-p)-2\zeta q][p(\zeta q-p)^2]^{-1}$, is tabulated for some values of p in Table 1. As a function of p, the proportion of uncensored observations, this ARE is monotone decreasing, goes to zero as p tends to 1 and to ∞ as p tends to zero. Interestingly, when p

is as large as 0.7, the ARE of the test T_1, which is based on just the failure counts, is higher than the test T_2 which utilizes the data of failure times. However, for p close to 1, i.e. when the censoring is very light, T_2 performs substantially better than T_1.

In a nonparametric setting, the locally most powerful rank test (LMPRT) for exponential scale alternatives is based on the statistic

$$S_N = \sum_{i=1}^{r} Z_i\beta_i + \frac{m-m_r}{N-r}\sum_{r+1}^{N} \beta_i \tag{5.2}$$

where β_i is given by (3.8) (see for example Basu (1968) or Johnson and Mehrotra (1972)). In order to compare the nonparametric tests T_1 and S, note that the efficacy of S is given by $e(S) = \lambda(1-\lambda)h^2 p$. Consequently, $ARE(T_1:S) = \zeta^2 q/p^2$ whose numerical values are given in Table 1. For p as large as 0.6 this ARE is larger than 93%. This shows that with moderate to heavy censoring the simple nonparametric test T_1 performs almost as well as the optimal rank test. Numerical values of ARE $(T_2:S)$ are also given in Table 1 and we see that T_2 does not outperform the LMPRT even for larger values of p. But this is not surprising in view of the asymptotic sufficiency of ranks under combined type II censoring which was established in Mehrotra and Johnson (1976). In fact, $ARE(T_2:S)$ decreases with heavier censoring and goes to zero as $p \to 0$.

TABLE 1

ASYMPTOTIC RELATIVE EFFICIENCIES

p	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.99
$ARE(T_1:T_2)$	1.309	1.281	1.248	1.209	1.160	1.098	1.015	0.897	0.695	0.218
$ARE(T_1:S)$	0.999	0.996	0.989	0.978	0.961	0.933	0.887	0.809	0.654	0.216
$ARE(T_2:S)$	0.763	0.777	0.792	0.809	0.828	0.849	0.874	0.904	0.941	0.992

Since considerations of sufficiency and invariance have led to the pair (T_1,T_2), the question naturally arises as to whether there is a combination of these components which is asymptotically as efficient as S. For this purpose, we seek constants c_1 and c_2 such that the test based on $(c_1T_1+c_2T_2)$ attains the maximum efficacy irrespective of a chosen sequence of local alternatives. From Corollary 1, the asymptotic distribution of $(c_1T_{1N}+c_2T_{2N})$, under the sequence of alternatives $\Delta_N = 1+h_N N^{-1/2}$, $h_N \to h$ as $N\to\infty$, is normal $\eta(-\lambda(1-\lambda)h^2\underset{\sim}{c}'\underset{\sim}{\mu}, \lambda(1-\lambda)\underset{\sim}{c}'\underset{\sim}{\tau}\underset{\sim}{c})$ where $\underset{\sim}{c}' = (c_1,c_2)$ and $\underset{\sim}{\mu}$ and $\underset{\sim}{\tau}$ are given by (4.4). The efficacy of this statistic is given by

$$e_{\underset{\sim}{c}} = \lambda(1-\lambda)h^2(\underset{\sim}{c}'\underset{\sim}{\mu})^2/\underset{\sim}{c}'\underset{\sim}{\tau}\underset{\sim}{c} \;. \tag{5.3}$$

By Cauchy-Schwarz inequality, the ratio $(\underset{\sim}{c}'\underset{\sim}{\mu})^2/\underset{\sim}{c}'\underset{\sim}{\tau}\underset{\sim}{c}$ attains its maximum value $\underset{\sim}{\mu}'\underset{\sim}{\tau}^{-1}\underset{\sim}{\mu}$ at $\underset{\sim}{c}^* = \underset{\sim}{\tau}^{-1}\underset{\sim}{\mu}$. Using (4.4) and after some simplifications $\underset{\sim}{c}^*$ reduces to $(1,-1)'$ and $\underset{\sim}{\mu}'\underset{\sim}{\tau}^{-1}\underset{\sim}{\mu} = p$. Therefore, the test based on $T = T_1 - T_2$ provides the asymptotically best invariant test in the sense of maximizing the Pitman ARE. Its efficacy is

$$e(T) = \lambda(1-\lambda)h^2 p \tag{5.4}$$

which is the same as that of the IMPRT.

A structural relation between the parametric test statistic T and the rank statistic S is evidenced through the following heuristic considerations. First note that under the null hypothesis $E(W_i) = \theta^{-1}\beta_i$, $i = 1,2,\ldots,N$, $E(U) = \theta^{-1}r$ where θ is the common scale parameter. Also, we have the relation $(N-r)^{-1}\sum_{i=r+1}^{N}\beta_i = 1+\beta_r$, and consequently we can write (5.2) as

$$S_N = \frac{r\left\{\sum_{i=1}^{r} Z_i E(W_i) + (m-m_r)E(W_r)\right\}}{E(U)} - m_r + m$$

$$= r\left[\frac{E(U_1|\underset{\sim}{Z})}{E(U)} - \frac{m_r}{r}\right] + m \,. \tag{5.5}$$

Therefore, one could view T_N as obtained from $(S_N - m)r^{-1}$ by simply removing the expectation in (5.5).

Finally we relate T_N to the likelihood ratio test. Since, the maximum likelihood estimators of θ_1 and θ_2 obtained from (2.2) are $\hat{\theta}_1 = U_1^{-1}M_r$ and $\hat{\theta}_2 = U_2^{-1}N_r$ whereas the common parameter θ, under the null hypothesis is estimated by $\hat{\theta} = U^{-1}r$, the likelihood ratio test statistic is given by $\Lambda_N = \hat{\theta}^r/(\hat{\theta}_1^{m_r}\hat{\theta}_2^{n_r})$. In terms of T_{1N} and T_{2N}, Λ_N can be expressed as

$$\Lambda_N = \left(\frac{T_{2N}}{T_{1N}}\right)^{rT_{1N}} \left(\frac{1-T_{2N}}{1-T_{1N}}\right)^{r(1-T_{1N})} \,. \tag{5.6}$$

Letting $\eta(T_{1N},T_{2N}) = -2N^{-1}\ln\Lambda_N$, it is easily seen that

$\eta(T_{1N},T_{2N})$ and its first partial derivatives with respect to T_{1N} and T_{2N}, evaluated at $E(T_{1N},T_{2N}) = (\lambda_N,\lambda_N)$ are all zero. As a consequence of Corollary 1, it can be verified that under local alternatives each of $\partial^2\eta/\partial T_{1N}^2$, $-\partial^2\eta/\partial T_{1N}\partial T_{2N}$, $\partial^2\eta/\partial T_{2N}^2$ converges to $2p[\lambda(1-\lambda)]^{-1}$ in probability. Considering a Taylor expansion of $\eta(T_{1N},T_{2N})$ around (λ_N,λ_N)

$$N\eta(T_{1N},T_{2N}) = \frac{r}{\lambda_N(1-\lambda_N)} \, (T_{1N}-T_{2N})^2 + R_N ,$$

where the remainder R_N converges to 0 in probability. Thus the two-sided test based on T_N is asymptotically equivalent to the likelihood ratio test.

6. CONCLUDING REMARKS

The sum total of the times that a sample of units spend in a life testing experiment, called the total time on test, plays a significant role in inference problems associated with exponential reliability models. When testing equality of scale parameters of two exponential populations, the ratio (T_2) of the total time on test of the first sample to that of the combined sample has several attractive features when the samples are either fully observed or independently type II censored. Not only does it provide the exact optimal tests and confidence intervals but the simplicity of its distribution (beta) facilitates computation of significance points and power. However, the advantages of T_2 are largely eroded when the samples are jointly type II censored. Its null distribution

becomes considerably involved and the non-null distribution does not seem to be analytically tractable. More seriously, its ARE compared to the modified Savage rank test (S) is always less than 1 and decreases with heavier amounts of censoring. Consequently, the use of T_2 has very little to recommend in the case of joint type II censoring.

The test (T_1) based on failure counts has surprisingly small loss of efficiency relative to S when censoring is moderate to heavy. The extreme simplicity of its null distribution, ease of handling its non-null distribution and computation of power make it an attractive competitor of the locally most powerful rank test S for practical applications. Moreover, it requires a simpler data base, the failure counts, rather than the actual failure times as required by T_2 or their ranks as required by S. The test based on (T_1-T_2) emerges as the asymptotically optimal parametric test under the exponential model. While it recovers the loss of ARE of T_2, its exact distribution theory is even more complicated than T_2, and there is little to recommend it over the rank test S.

For the problem of testing a more general hypothesis $H_0 : \theta_2/\theta_1 = \Delta_0$, not necessairly 1, or the related problem of setting confidence intervals for Δ, the usefulness of rank test is severely limited when the samples are jointly censored. A reduction of the problem of testing $H_0 : \Delta = \Delta_0$ to that of testing $H_0 : \Delta = 1$ by rescaling one of the samples is a convenient trick that works for complete samples as well as separate type II censoring. But it does not apply in joint type II censoring, because such a rescaling

alters the ranks in an undetermined manner. An investigation of these problems is underway and the results will be reported in a separate article.

APPENDIX

An important ingredient in the proof of Theorem 1 is the use of an asymptotic expansion due to Sen (1968) for a sample quantile in the case of sampling from non-identical distributions. Let g_N denote the pdf of the mixture $G_N = \lambda_N F_{\Delta_0} + (1-\lambda_N)F_{\Delta_N}$, and let ζ_N and ξ_N respectively denote the p-quantiles of F_{Δ_N} and G_N. Since W_r is the p-quantile of the combined sample, and G_N is the corresponding mixture of the populations F_{Δ_0} and F_{Δ_N}, Theorem 2.1 of Sen (1968) entails that

$$Q_N \equiv W_r - \xi_N = [g_N(\xi_N)]^{-1}[p - H_N(\xi_N) + R_N] \qquad (A.1)$$

where H_N is the combined sample empirical distribution function, $R_N = O(N^{-3/4}\ell nN)$, a.s., and $N^{1/2}Q_N$ has a limiting normal distribution with mean zero. Using the simplified notations $F(x)$ and $f(x)$ for $F_{\Delta_0}(x)$ and $f_{\Delta_0}(x)$ respectively, let

$$b = \frac{g}{f(\zeta)}, \quad \nu_{1N} = \int_0^{\xi_N} \bar{F}(x)dx, \quad \nu_{2N} = \int_0^{\xi_N} \bar{F}_{\Delta_N}(x)dx, \quad \nu_{0N} = F(\xi_N). \qquad (A.2)$$

The principal steps involved in the proof of Theorem 1 are organized in a few lemmas.

<u>Lemma A1</u>: Let Z_{1N} and Z_{2N} be defined by

$$N^{-1/2}Z_{1N} = \int_0^{\xi_N} xd(F_{1m}-F) - (\zeta+\lambda b)[F_{1m}(\xi_N)-F(\xi_N)] - (1-\lambda)b[F_{2N}(\xi_N)-F_{\Delta_N}(\xi_N)]$$

$$N^{-1/2}Z_{2N} = \int_0^{\xi_N} xd(F_{2n}-F_{\Delta_N}) - \lambda b[F_{1m}(\xi_N)-F(\xi_N)] - [\zeta+(1-\lambda)b][F_{2n}(\xi_N)-F_{\Delta_N}(\xi_N)]. \qquad (A.3)$$

Then $N^{1/2}(U_1/m - \nu_{1N}) = Z_{1N} + o_p(1)$ and $N^{1/2}(U_2/n - \nu_{2N}) = Z_{2N} + o_p(1)$.

<u>Proof</u>: Referring to the expression for U_1 given in (4.1), integration by parts and use of the expansion (A.1) provide

$$\frac{U_1}{m} = \int_0^{W_r} \bar{F}_{1m}(x)dx = \int_0^{\xi_N} \bar{F}_{1m}(x)dx + \int_{\xi_N}^{W_r} \bar{F}_{1m}(x)dx$$

$$= \int_0^{\xi_N} xdF_{1m}(x) + \xi_N\bar{F}_{1m}(\xi_N) + Q_N\bar{F}(\xi_N) - \int_{\xi_N}^{\xi_N+Q_N} [F_{1m}(x)-F(\xi_N)]dx. \qquad (A.4)$$

Denoting the last integral in the rhs of (A.4) by C_N, we have

$$N^{1/2}|C_N| \le N^{1/2}|Q_N|\left\{\sup_x|F_{1m}(x)-F(x)| + [F(\xi_N+|Q_N|) - F(\xi_N)]\right\}.$$

Since $N^{1/2}Q_N$ has a limit distribution, F is continuous, $\xi_N \to \zeta$, and since $\sup|F_{1m}-F| \to 0$ a.s. by Glivenko-Cantelli Theorem, we have $N^{1/2}C_N \xrightarrow{P} 0$. Using (A.1), (A.2) and (A.4) we have after some rearrangement of terms

$$\frac{U_1}{m} - \nu_{1N} = \int_0^{\xi_N} x\,d(F_{1m}-F) - [\xi_N + \lambda_N \bar{F}(\xi_N)(g_N(\xi_N))^{-1}][F_{1m}(\xi_N)-F(\xi_N)]$$

$$- (1-\lambda_N)\bar{F}(\xi_N)(g_N(\xi_N))^{-1}[F_{2N}(\xi_N)-F_{\Delta_N}(\xi_N)] + o_p(N^{-1/2}) .$$

In view of the limits $\lambda_N \to \lambda$, $\bar{F}(\xi_N) \to \bar{F}(\zeta) = q$ and $g_N(\xi_N) \to f(\zeta)$, the result stated for U_1 follows. The proof for U_2 is analogous. ▌

<u>Lemma A2</u>: With ν_{ON} defined in (A.2), $N^{1/2}(M_r/m - \nu_{ON}) = Z_{ON} + o_p(1)$ where

$$N^{-1/2} Z_{ON} = (1-\lambda)\Big\{[F_{1m}(\xi_N)-F(\xi_N)] - [F_{2n}(\xi_N)-F_{\Delta_N}(\xi_N)]\Big\} . \qquad \text{(A.5)}$$

<u>Proof</u>: Letting $C_{ON} = F_{1m}(\xi_N+Q_N) - F_{1m}(\xi_N) - Q_N f(\xi_N)$ and referring to expression (4.1) of M_r, we have from (A.1) and (A.2)

$$\left(\frac{M_r}{m} - \nu_{ON}\right) = \left[1-\lambda_N \frac{f(\xi_N)}{g_N(\xi_N)}\right]\left[F_{1m}(\xi_N)-F(\xi_N)\right] - (1-\lambda_N)[F_{2n}(\xi_N)-F_{\Delta_N}(\xi_N)] + C_{ON} .$$

Since $f(\xi_N)/g_N(\xi_N) \to 1$, the stated result would follow once we establish that $N^{1/2} C_{ON} \xrightarrow{P} 0$. A somewhat similar result was established in Theorem 2 of Ghosh (1971) in context of a single random sample. The same approach will be used here with necessary adaptation of the technical details.

With an arbitrary but fixed t, define $p_{N,t} = F(\xi_N) - tm^{-1/2}$ and note that $p_{N,t} \to F(\zeta) = p$ as $N \to \infty$. For large enough m, define $\tilde{X}_m$ to be the p_N-quantile of the sample $\{X_1, \ldots, X_m\}$, so we have the representation

$$\tilde{X}_m = \zeta + \{f(\zeta)\}^{-1}[p_{N,t} - F_{1m}(\zeta)] + R_m, \quad \sqrt{m}\,R_m \xrightarrow{P} 0 . \qquad \text{(A.6)}$$

Defining

$$\tilde{V}_m = m^{1/2}[F(\xi_N) - F_{1m}(\xi_N+Q_N)]$$

$$\tilde{W}_m = m^{1/2}[F(\xi_N) - F_{1m}(\xi_N) - Q_N f(\xi_N)]$$

$$D_m = m^{1/2}[F(\xi_N) - F_{1m}(\zeta) - f(\zeta)(\xi_N - \zeta + Q_N)]$$

we have $m^{1/2} C_{ON} = \tilde{W}_m - \tilde{V}_m$. Now the event relations

$$[\tilde{V}_m \le t] = [p_{N,t} \le F_{1m}(\xi_N+Q_N)] = [\tilde{X}_m \le \xi_N + Q_N] ,$$

and the representation (A.6) entail $[\tilde{V}_m \le t] = [f(\zeta)\sqrt{m}\,R_m + D_m \le t]$. But D_m can be expressed as $D_m = \tilde{W}_m + \tilde{Z}_{1m} + \tilde{Z}_{2m}$ where

$$\tilde{Z}_{1m} = m^{1/2} Q_N [f(\xi_N) - f(\zeta)]$$

$$\tilde{Z}_{2m} = m^{1/2}[F_{1m}(\xi_N) - F_{1m}(\zeta) - f(\zeta)(\xi_N - \zeta)] .$$

Since $m^{1/2} Q_N$ has a limit distribution and $\xi_N \to \zeta$, we have $\tilde{Z}_{1m} \xrightarrow{P} 0$. Also $\tilde{Z}_{2m} \xrightarrow{P} 0$ because $E\tilde{Z}_{2m} \to 0$ and $\operatorname{var} \tilde{Z}_{2m} \to 0$. Therefore, we have an event identity $[\tilde{V}_m \le t] = [\tilde{W}_m + \tilde{Z}_m \le t]$ where $\tilde{Z}_m = f(\zeta) m^{1/2} R_m + \tilde{Z}_{1m} + \tilde{Z}_{2m} \xrightarrow{P} 0$. As a consequence of Lemma A.3 below, $\tilde{W}_m$ has a limiting normal distribution. Therefore, given $\epsilon > 0$, there exists δ and m_0 such that $P[|\tilde{W}_m| > \delta] < \epsilon$ for all $m > m_0$. Using Lemma 1 of Ghosh (1971), we then conclude that $\tilde{V}_m - \tilde{W}_m \xrightarrow{P} 0$. ▌

Lemma A3: Let $Z^*_{1N} = Z_{1N} - Z_{2N}$ and $Z^*_{0N} = \lambda Z_{0N}$. Then the limiting distribution of (Z^*_{0N}, Z^*_{1N}) is bivariate normal $\mathcal{N}_2(\underset{\sim}{0}, \underset{\sim}{\Sigma}^*)$ where

$$\underset{\sim}{\Sigma}^* = \begin{pmatrix} \lambda(1-\lambda)\sigma_{11} & \sigma_{12} - \zeta\sigma_{11} \\ \sigma_{12} - \zeta\sigma_{11} & [\lambda(1-\lambda)]^{-1}(\sigma_{22} - 2\zeta\sigma_{12} + \zeta^2\sigma_{11}) \end{pmatrix} \quad (A.7)$$

and σ_{ij}'s are given in (4.2).

Proof: Letting $I(a,b) = 1(0)$ if $a \le (>) b$, we define the random vector $\underset{\sim}{V}'_N = (V_{1N}, V_{2N}, V_{3N}, V_{4N})$ by

$$\begin{aligned} V_{1N} &= m^{1/2}[F_{1m}(\xi_N) - F(\xi_N)] = m^{-1/2} \sum_{i=1}^{m} [I(X_i, \xi_N) - F(\xi_N)] \\ V_{2N} &= m^{1/2} \int_0^{\xi_N} x\, d(F_{1m} - F) = m^{-1/2} \sum_{i=1}^{m} [X_i I(X_i, \xi_N) - \int_0^{\xi_N} x\, dF] \\ V_{3N} &= n^{1/2}[F_{2n}(\xi_N) - F_{\Delta_N}(\xi_N)] = n^{-1/2} \sum_{j=1}^{n} [I(Y_j, \xi_N) - F_{\Delta_N}(\xi_N)] \\ V_{4N} &= n^{1/2} \int_0^{\xi_N} x\, d(F_{2n} - F_{\Delta_N}) = n^{-1/2} \sum_{j=1}^{n} [Y_j I(Y_j, \xi_N) - \int_0^{\xi_N} y\, dF_{\Delta_N}] . \end{aligned} \quad (A.8)$$

Each $V_{\alpha N}$ consists of a sum of iid bounded random variables with mean 0. The pairs (V_{1N}, V_{2N}) and (V_{3N}, V_{4N}) are independent. Noting that the covariance matrix of $[I(X_i, \Delta_0), X_i I(X_i, \Delta_0)]$ is precisely $\underset{\sim}{\Sigma}$ defined in (4.2), it is easy to verify that under Assumptions (A) the covariance matrix of each of the subvectors (V_{1N}, V_{2N}) and (V_{3N}, V_{4N}) tend to the limit $\underset{\sim}{\Sigma}$ as $N \to \infty$. By multivariate central limit theorem, the limit distribution of $\underset{\sim}{V}_N$ is $\mathcal{N}_4(\underset{\sim}{0}, \underset{\sim}{\Sigma}_0)$ where

$$\underset{\sim}{\Sigma}_0 = \begin{pmatrix} \underset{\sim}{\Sigma} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{\Sigma} \end{pmatrix} .$$

From (A.3), (A.5) and (A.8), we relate Z^*_{0N} and Z^*_{1N} in terms of $\underset{\sim}{V}_N$ and note that the limit distribution of $(Z^*_{0N}, Z^*_{1N})'$ is the same as that of $\underset{\sim}{L}\underset{\sim}{V}_N$ where

$$\underset{\sim}{L} = \begin{pmatrix} 0 & \sqrt{\lambda}(1-\lambda) & 0 & -\lambda\sqrt{1-\lambda} \\ \frac{1}{\sqrt{\lambda}} & -\frac{\zeta}{\sqrt{\lambda}} & -\frac{1}{\sqrt{1-\lambda}} & \frac{\zeta}{\sqrt{1-\lambda}} \end{pmatrix}$$

and the latter is $\mathcal{N}_2(\underset{\sim}{0}, \underset{\sim}{L}\underset{\sim}{\Sigma}_0\underset{\sim}{L}')$. That the matrix product $\underset{\sim}{L}\underset{\sim}{\Sigma}_0\underset{\sim}{L}'$ reduces to $\underset{\sim}{\Sigma}^*$ can be conveniently seen by carrying out the multiplication in the partitioned form $\underset{\sim}{L}\underset{\sim}{\Sigma}_0\underset{\sim}{L}' = \underset{\sim}{L}_1\underset{\sim}{\Sigma}\underset{\sim}{L}'_1 + \underset{\sim}{L}_2\underset{\sim}{\Sigma}\underset{\sim}{L}'_2$ where $\underset{\sim}{L} = (\underset{\sim}{L}_1 \vdots \underset{\sim}{L}_2)$. ∎

Proof of Theorem 1: Recalling that $T_{1N} = M_r/r$ and using Lemma A.2 we have

$$\begin{aligned} N^{1/2}(T_{1N} - \lambda) &= N^{1/2} \frac{m}{r}\left(\frac{M_r}{m} - \nu_{0N}\right) + N^{1/2}\left(\frac{m}{r}\nu_{0N} - \lambda\right) \\ &= p^{-1}\lambda Z_{0N} + N^{1/2}\left(\frac{m}{r}\nu_{0N} - \lambda\right) + o_p(1) . \end{aligned} \quad (A.9)$$

Since

$$\lambda_N \nu_{1N} + (1-\lambda_N)\nu_{2N} = \int_0^{\xi_N} \bar{G}_N(x)\, dx$$

where $G_N(x) \to F(x)$, we have from Lemma A1, $N^{-1}(U_1+U_2) = \lambda_N \nu_{1N} + (1-\lambda_N)\nu_{2N} + o_p(1)$, so

$$N^{-1}(U_1+U_2) \xrightarrow{P} \int_0^\zeta \bar{F}(x)dx = \int_0^\zeta x dF(x) + \zeta\bar{F}(\zeta) = \nu . \tag{A.10}$$

Again using Lemma A1 along with (A.10), we have

$$\begin{aligned} N^{1/2}(T_{2N}-\lambda) &= N^{1/2}\Big[\lambda_N(1-\lambda)\frac{U_1}{m} - \lambda(1-\lambda_N)\frac{U_2}{n}\Big]\Big[(U_1+U_2)/N\Big]^{-1} \\ &= \frac{\lambda(1-\lambda)}{\nu}\Big[(Z_{1N}-Z_{2N}) + N^{1/2}\Big\{\frac{\lambda_N}{\lambda}\nu_{1N} - \frac{(1-\lambda_N)}{(1-\lambda)}\nu_{2N}\Big\}\Big] + o_p(1) . \end{aligned} \tag{A.11}$$

Lemma A3 implies a limiting bivariate normal distribution of $p^{-1}\lambda Z_{0N}$ and $\nu^{-1}\lambda(1-\lambda)(Z_{1N}-Z_{2N})$ with mean $\underset{\sim}{0}$ and covariance matrix $\underset{\sim}{\Sigma}$ given in (4.3). To complete the proof, it remains only to establish the limits

$$\begin{aligned} &\lim N^{1/2}\Big(\frac{m}{r}\nu_{0N}-\lambda\Big) = -h\lambda(1-\lambda)p^{-1}\overset{o}{F}(\zeta) \\ &\lim N^{1/2}\Big(\frac{\lambda_N}{\lambda}\nu_{1N} - \frac{1-\lambda_N}{1-\lambda}\nu_{2N}\Big) = h\int_0^\zeta \overset{o}{F}(x)dx . \end{aligned} \tag{A.12}$$

To derive the first result, note that $F_{\Delta_0}(\zeta) = p = F_{\Delta_N}(\zeta_N)$, so

$$N^{1/2}[F_{\Delta_N}(\zeta_N)-F_{\Delta_N}(\zeta)] = -N^{1/2}[F_{\Delta_N}(\zeta)-F_{\Delta_0}(\zeta)] . \tag{A.13}$$

The rhs of (A.13) tends to the limit $-h\overset{o}{F}(\zeta)$ and considering the limit of the ℓhs, we obtain $\lim N^{1/2}(\zeta_N-\zeta) = -h\overset{o}{F}(\zeta)/f(\zeta)$. On the other hand, considering limit on each side of the identity

$$\lambda_N[F_{\Delta_0}(\xi_N)-F_{\Delta_0}(\zeta)] = (1-\lambda_N)[F_{\Delta_N}(\zeta_N)-F_{\Delta_N}(\xi_N)] ,$$

we obtain the relation $\lim N^{1/2}(\xi_N-\zeta) = (1-\lambda)\lim N^{1/2}(\zeta_N-\zeta)$. These results along with the assumption $N^{1/2}(\lambda_N-\lambda) \to 0$ and $r = [Np]$ imply the first statement in (A.12). As for the second, it is enough to consider the limit of $N^{1/2}(\nu_{1N}-\nu_{2N}) = N^{1/2}\int_0^{\xi_N}[F_{\Delta_N}(x)-F_{\Delta_0}(x)]dx$. Since $N^{1/2}(\Delta_N-\Delta_0) \to h$, Assumption (A3) entails the stated result. ∎

REFERENCES

Bain, Lee J. (1978), Statistical Analysis of Reliability and Life-testing Models--Theory and Methods, New York: Marcel Dekker, Inc.

Basu, Asit P. (1968), "On a generalized Savage statistic with applications to life testing," Annals of Mathematical Statistics, 39, 1591-1604.

Epstein, B. and Sobel, M. (1953), "Life testing," Journal of the American Statistical Association, 48, 486-502.

Epstein, B. and Tsao, C. K. (1953), "Some tests based on ordered observations from two exponential populations," Annals of Mathematical Statistics, 24, 458-466.

Gehan, E. A. (1965), "A generalized Wilcoxon test for comparing arbitrarily singly-censored samples," Biometrika, 52, 203-223.

Ghosh, J. K. (1971), "A new proof of the Bahadur representation of quantiles and an application," Annals of Mathematical Statistics, 42, 1957-1961.

Halperin, Max (1960), "Extension of the Wilcoxon-Mann-Whitney test to samples censored at the same fixed point," Journal of the American Statistical Association, 55, 125-138.

Hájek, J. and Šidák (1967), Theory of Rank Tests, New York: Academic Press.

Johnson, Richard A. (1974), "Some optimality results for a one and two sample procedures based on smallest r order statistics," Reliability and Biometry-Statistical Analysis of Lifelength, Editor Proschan and Serfling, Philadelphia: SIAM, 459-493.

Johnson, Richard A. and Mehrotra, Kishan G. (1972), "Locally most powerful rank tests for the two-sample problem with censored data," Annals of Mathematical Statistics, 43, 823-831.

Mehrotra, Kishan G. and Johnson, Richard A. (1976), "Asymptotic sufficiency and asymptotically most powerful tests for the two sample censored situation," The Annals of Statistics, 4, 589-596.

Perng, S. K. (1977), "An asymptotically efficient test for the location parameter and the scale parameter of an exponential distribution," Communications in Statistics--Theory and Methods, A6, 1399-1407.

Rao, V., Savage, I. R. and Sobel, M. (1960), "Contribution to the theory of rank order statistics: The two sample censored case," Annals of Mathematical Statistics, 31, 415-426.

Sen, Pranab K. (1968), "Asymptotic normality of sample, quantiles for m-dependent processes," Annals of Mathematical Statistics, 39, 1724-1730.

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The problem of testing equality of the scale parameters of two exponential distributions is considered under a combined sample type II censoring scheme that has been extensively treated in nonparametric inference. Sufficiency and invariance considerations lead to a pair of statistics, namely, the proportions of failure count and the total time on test of the first sample to those of the combined sample. Exact distribution and moment properties are discussed, and asymptotic joint normality under local alternatives is established in a general framework. The invariant test that maximizes the Pitman efficacy is seen to be based on the difference between these two proportions, and is asymptotically equivalent to the likelihood ratio test.

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